① Sets in $\mathbb{C}$:

- The symbol $\subset$ will allow for equality of sets. (Notation).
- $D(a,R) = \{z \mid 0 \le |z-a| < R\}$ is the open disc of radius R centered at point a. $D(a,R)\setminus\{a\}$ is a punctured open disc.
- A neighbourhood of a is an open disc of nonzero radius centered at a. Similarly for a punctured neighbourhood.

Important types of sets.

$S \subset \mathbb{C}$ is open $\iff (S=\emptyset) \vee (\forall a \in S)(\exists \varepsilon>0)(D(a,\varepsilon)\subset S)$.

The compliment of a set is $\mathrm{comp}(S) = \mathbb{C}\setminus S$

S is closed $\iff \mathrm{comp}(S)$ is open

$z \in \partial S$ (the boundary of S) $\iff (\forall R>0)(\exists s,s' \in D(z,R)$ such that $s \in S$ & $s' \in \mathrm{comp}(S))$

Its important to remember that $\emptyset$ & $\mathbb{C}$ are the only sets BOTH open & closed, however there are many sets that are neither.

A nonempty $S \subset \mathbb{C}$ is connected if any two points from S can be connected by a continuous path. S is not connected $\iff$ S is disconnected.

A domain is a nonempty, connected, open set. Only a set term, not domain of a function

$S \subset \mathbb{C}$ is bounded $\iff (\exists R>0)(S \subset D(0,R))$.

S is compact $\iff$ S is closed, bounded.

Point $a \in \mathbb{C}$ is a point of accumulation for $S \subset \mathbb{C}$
$\iff (\forall \varepsilon>0)(\exists s \in S)(s \in D(a,\varepsilon)\setminus\{a\}.)$

② Sequences & Limits:

A complex sequence is an ordered subset of points $\{u_j\} \subset \mathbb{C}$.

The sequence $\{u_n\}_{n\in\mathbb{N}}$ can be said to:

- Converge to $u \iff \lim_{n\to\infty} u_n = u \iff (\forall \varepsilon>0)(\exists N \in \mathbb{N})(\forall m>N)(u_m \in D(u,\varepsilon))$
- Diverge $\iff (\forall u \in \mathbb{C})(\lim_{n\to\infty} u_n \neq u) \iff$ NOT convergent
- Diverge to $\infty \iff (\forall K>0)(\exists N>0)(\forall m>N)(u_m \in \mathrm{Comp}[D(0,K)])$

Sequence $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy $\iff (\forall \varepsilon>0)(\exists N>0)(\forall m,n>N)(|u_m-u_n|<\varepsilon)$

Limit Rules. $u_n \to u$ & $v_n \to v$ THEN

- $u_n+v_n \to u+v$
- $\forall \lambda \in \mathbb{C}$ $\lambda u_n \to \lambda u$
- $u_n v_n \to uv$
- $\frac{u_n}{v_n} \to \frac{u}{v}$, when $v, v_1, v_2, \dots \neq 0$.

Convergence Theorems.

T: $u_n \to \infty$ in $\mathbb{C} \iff \frac{1}{u_n} \to 0$

T: $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ converges $\iff \{\mathrm{Re}(u_n)\}$ & $\{\mathrm{Im}(u_n)\}$ converge

The complex sequence converges iff its two components converge.

T: A sequence converges $\iff$ The sequence is Cauchy.

T: Every bounded sequence has a convergent subsequence.
(Bolzano – Weierstrass Theorem)

③ Continuity & Limits of Functions:

For $S \subset \mathbb{C}$ open. $f: S \longrightarrow \mathbb{C}$.

- $\lim_{z\to c} f(z) = L \iff (\forall \varepsilon>0)(\exists \delta>0)(z \in D(c,\delta)\setminus\{c\} \Rightarrow f(z) \in D(L,\varepsilon))$
- $\lim_{z\to\infty} f(z) = L \iff (\forall \varepsilon>0)(\exists K>0)(z \in \mathrm{comp}(D(0,K)) \cap S \Rightarrow f(z) \in D(L,\varepsilon))$.
- f is continuous at $c \in \mathbb{C} \iff \lim_{z\to c} f(z) = f(c)$
- f is continuous in $S \iff (\forall s \in S)($ f is continuous at $s)$.

T: f continuous at c & $f(c) \neq 0 \implies (\exists \varepsilon>0)(\forall s \in D(c,\varepsilon))(f(s)\neq 0)$.

If f is continuous & nonzero at a point then there is a neighbourhood in around that point in which f is also nonzero.

Limit Rules. Assuming $\lim_{z\to c} f(z)$ & $\lim_{z\to c} g(z)$ exist.

- $\lim_{z\to c}[f(z)+g(z)] = \lim_{z\to c} f(z) + \lim_{z\to c} g(z)$
- Same for product, quotient, & composition.

④ The Basics of Holomorphicity:

$S \subset \mathbb{C}$ open. $f: S \longrightarrow \mathbb{C}$ is complex differentiable at $c \in S \iff$ The limit exists $f'(c) = \frac{f(z)-f(c)}{z-c}$

f is complex differentiable in S if it has a derivative at every point. f' is the derivative.

T: $(f+g)' = f'+g'$, $(fg)' = f'g+fg'$, $(f\circ g)' = (f'\circ g)g'$
Assuming all appropriate limits exist.

T: complex differentiable $\Rightarrow \mathbb{R}^2$ diff

T: Existence & continuity of partial derivatives in $\mathbb{R}^2$ is sufficient for differentiability ($\mathbb{R}^2$ diff) at that point.

Holomorphic. function f is holomorphic at $c \in \mathbb{C} \iff (\exists \varepsilon>0)(f$ is complex differentiable in $D(c,\varepsilon))$.

f holomorphic in open set $S \iff$ holomorphic at all $s \in S$.

f is entire $\iff$ f is holomorphic on all of $\mathbb{C}$.

T: Let $c=a+ib$ & $z=x+iy$, $f(z)=F(x,y) = u(x,y)+iv(x,y)$

Where u & v are real functions. Then we can say

f is holomorphic at $c \iff$
- u & v are $\mathbb{R}^2$ diff in a neighbourhood of (a,b).
- $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$

The partial derivatives must be related by Cauchy – Riemann relations

T: $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

T: f holomorphic in open $S \subset \mathbb{C} \implies f'$ continuous on S.

T: f holomorphic in Ω a domain & $|f| = c \in \mathbb{C}$ in Ω $\implies f' = 0$ in Ω.

{ Derivatives & entire functions }

⑤ Curves in $\mathbb{C}$:

- A continuous curve is a function $\gamma:[a,b]\longrightarrow\mathbb{C}$ that is continuous
- γ is a simple curve $\iff [\gamma(t_1)=\gamma(t_2)\iff t_1=t_2]$
- γ is a closed curve $\iff \gamma(a)=\gamma(b)$
- γ simple closed curve $\iff [(\forall t_1<t_2)(\gamma(t_1)=\gamma(t_2)\iff t_1=a \,\&\, t_2=b)]$
- $\gamma(t)=\xi(t)+i\eta(t)$, $t\in[a,b]$ is a regular arc if both ξ & η are differentiable on $[a,b]$ and $\gamma'(t)=\xi'(t)+i\eta'(t)$ is continuous & nonzero on (a,b).

T: A simple closed curve C divides the complex plain into two domains, I & E, where one is bounded & the other not. C is the boundary of both I & E.

T: A regular arc has a finite length given by
$$L=\int_a^b|\gamma'(t)|dt=\int_a^b\sqrt{\xi'(t)^2+\eta'(t)^2}\,dt$$

⑥ Contour Integrals:

The contour integral of a complex function f over a regular arc C is is given by $\int_C f(z)dz=\int_a^b f(\gamma(t))\gamma'(t)dt$ Where $\gamma(t)$ is a parametrisation of C on $[a,b]$.

T: f holomorphic on a regular arc γ $\implies g=f\circ\gamma$ is a differentiable function of real valued t & $g'(t)=f'(\gamma(t))\gamma'(t)$

T: f holomorphic in domain $\Omega \implies [f'=0$ in $\Omega\iff f=c\in\mathbb{C}$ in $\Omega]$

T: Contour integrals are linear maps of functions into $\mathbb{C}$, ie.
$$\int_C(f+g)(z)dz=\int_C f(z)dz+\int_C g(z)dz$$
$$(\forall\alpha\in\mathbb{C})\left(\int_C\alpha f(z)dz=\alpha\int_C f(z)dz\right)$$

T: f continuous on $[\alpha,\beta]\implies\left|\int_\alpha^\beta f(t)dt\right|\le\int_\alpha^\beta|f(t)|dt$

A contour C is a finite number of regular arcs joined end to end. $\int_C f(z)dz=\int_{C_1}f(z)dz+\dots+\int_{C_n}f(z)dz$

T: $f(z)=F'(z)$ on contour C, starting at z_1, ending z_2 $\implies \int_C f(z)dz=F(z_2)-F(z_1)$ for some holomorphic F

T: $|f(z)|\le M$ on contour C of length L $\implies\left|\int_C f(z)dz\right|\le ML$

T: C a contour, Ω a domain disjoint from C. ϕ is absolutely integrable on C
$$\implies(\forall n\in\mathbb{N})\left(\Psi_n(z)=\int_C\frac{\phi(t)dt}{(t-z)^n}\text{ is holomorphic in }\Omega\text{ \& }\frac{d}{dz}\Psi_n(z)=n\Psi_{n+1}(z)\right)$$
$\implies\Psi_1$ has complex derivatives of all orders in Ω

When C is a closed contour we denote the integral around it by $\oint_C$.
$$\oint_C f(z)dz=-\oint_C f(z)dz$$

Changing Contours.

A domain Ω is starshaped if there is an $\ell\in\Omega$ such that for all $z\in\Omega$ the straight line joining ℓ to z lies inside Ω. ℓ is called the lookout point.

T: Cauchy's Thm: For any closed contour C in star domain Ω where f is holomorphic $\oint_C f(z)dz=0$.

T: Ω stardomain, f holomorphic in Ω, then for any two $C_1,C_2\subset\Omega$ with the same start and end points $\int_{C_1}f(z)dz=\int_{C_2}f(z)dz$ Deformations

T: f holomorphic in star domain Ω except at possibly z_0. For any contour $C\subset\Omega$ with z_0 in its interior we have
$$\oint_C f(z)dz=\oint_{\partial D(z_0,\rho)}f(z)dz$$
$\forall\rho$ such that $D(z_0,\rho)$ is in the interior of C.

T: Cauchy's Integral Formula: f holomorphic in domain $\Omega\subset\mathbb{C}$ and $C\subset\Omega$ simply closed contour
$$\implies\forall z\text{ in the interior of }C\quad f^{(n)}(z)=\frac{n!}{2\pi i}\oint_C\frac{f(t)dt}{(t-z)^n}$$

⑦ Moduli & Extrema:

For $S\subset\mathbb{C}$ open, a local max/min of $\varphi:S\to\mathbb{R}$ is a point $c\in S$ with $\varphi(z)\le\varphi(c)$ ($\ge$ for min) for any z in a neighbourhood of c.

A saddle point $c\in\mathbb{C}$ of a twice $\mathbb{R}^2$ diff $\varphi:S\to\mathbb{R}$ is a point c such that for $z=x+iy$
$$\frac{\partial\varphi}{\partial x}(c)=\frac{\partial\varphi}{\partial y}(c)=0\ \&\ \left[\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}-\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2\right]_{z=c}<0.$$

T: f holomorphic on Ω a domain $|f|$ attains a local max at some point in Ω $\implies f=c$ on Ω. a constant

The maximum modulus of a holomorphic function is attained on the boundary of the domain.

T: The max & min of the real & imaginary parts of holomorphic f on Ω are approached on $\partial\Omega$

T: Each critical point ($f'(c)=0$) on Ω of holomorphic f is a saddle point.

T: Cauchy's Inequality: f holomorphic on an open set containing $\partial D(z_0,R)\cup D(z_0,R)$ & $|f(z)|\le M$ $\forall z\in\partial D(z_0,R)$
$$\implies|f^{(n)}(z_0)|\le\frac{n!M}{R^n}$$

T: Liouville: All bounded entire functions are constant.

T: $P(z)$ noncanstant polynomial $\implies\exists z_0\in\mathbb{C}$ $P(z_0)=0$.

T: $(f$ entire$)(\exists A,B,\lambda>0)(\forall z\in\mathbb{C})(|f(z)|<A+B|z|^\lambda)$
$\implies f$ is polynomial degree $\le\lambda$.

T: $(f$ entire$)(\exists R,K>0)(|z|\ge R\Rightarrow|f(z)|\ge K)\implies f$ polynomial

⑧ Analytic Continuation I:

Taylor's Theorem.

f holomorphic in Ω a domain $\implies \forall z \in D(z_0,R) \subset \Omega$ The largest open disc centered on z_0, $R>0$.

$$f(z) = \sum_{j\geq 0} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$$

- This series is the taylor series. R is the radius of convergence.

A function that is representable by a taylor series in a neighbourhood of a point is called analytic, at that point. Analitic $\equiv$ Holomorphic on $\mathbb{C}$.

- The radius of convergence R is the distance from the point of expansion to the nearest non-holomorphic point.

T: $\frac{1}{R} = \lim_{j\to\infty} \sup_{\ell > j} \left| \frac{f^{(\ell)}(z_0)}{\ell!} \right|^{\frac{1}{\ell}}$

Continuations.

T: f & g holomorphic on domain Ω, $S \subset \Omega$ a closed set such that $\exists c \in \Omega$, c a point of accumulation for S.

$$(\forall z \in S)(f(z)=g(z)) \implies (\forall z \in \Omega)(f(z)=g(z)).$$

For f holomorphic in domain Ω & g defined on $S \subset \Omega$ with an accumulation point in Ω, then if $(f(z)=g(z))(\forall z \in S)$ we call f the analytic continuation of g to the domain Ω.

⑨ Zeros & Singularities:

f holomorphic in domain Ω

- z_0 a zero of f $\iff f(z_0)=0$
- z_0 an isolated zero of f $\iff f(z_0)=0$ AND $(\exists \varepsilon>0)(\forall z \in D(z_0,\varepsilon)\setminus\{z_0\})(f(z)\neq 0)$
- z_0 a zero of order $m \in \mathbb{N}$ $\iff f(z_0)=0$ and $\lim_{z\to z_0} \frac{f(z)}{(z-z_0)^m} = L \neq 0$. exists & is non zero

T: f holomorphic in domain Ω

$\implies$ ① · Every zero is isolated in Ω

· Every zero has a well defined order $\in \mathbb{N}$

· There are only finitely many zeroes in any compact subset of Ω

OR ② $(\forall z \in \Omega)(f(z)=0)$.

Singularities.

If f is holomorphic in $D(c,\varepsilon)\setminus\{c\}$ but not at c, then c is an isolated singularity of f. If there exists a constant k at c for which $g(z) = \begin{cases} f(z), & z\neq c \\ k, & z=c \end{cases}$ is holomorphic at c then c is a removable singularity of f.

T: L'Hopitals: For f & g with zeroes at $z=c$ of order m then,

$$\lim_{z\to c}\frac{f(z)}{g(z)} = \lim_{z\to c}\frac{f^{(m)}(z)}{g^{(m)}(z)}$$

$$\sum_{n=-\infty}^{-1} c_n = \sum_{n=1}^{\infty} c_{-n} \quad \& \quad \sum_{n=-\infty}^{\infty} c_n = \sum_{n=-\infty}^{-1} c_n + \sum_{n=1}^{\infty} c_n$$

T: Laurent's Theorem: If f has an isolated singularity at z_0 and is holomorphic inside $D(z_0,R)\setminus\{z_0\}$ THEN f is representable as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{2\pi i}\oint_{|t-z_0|=\rho} \frac{f(t)\,dt}{(t-z_0)^{n+1}}, \text{ some } \rho\in(0,R).$$

This series is the Laurent series of f.

The series $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is the principle part of the series.

If $\exists m \in \mathbb{N}$ such that the Laurent series is $\sum_{n=-m}^{\infty} a_n(z-z_0)^n$ then z_0 is a pole of order m. If there is no such m then z_0 is an isolated essential singularity.

$$\operatorname*{Res}_{z=z_0}\{f(z)\} = a_{-1} = \frac{1}{2\pi i}\oint_C \frac{f(t)\,dt}{(t-z_0)^{n+1}}\bigg|_{n=-1} = \frac{1}{2\pi i}\oint_C f(t)\,dt$$

A pole of order $m\in\mathbb{N}$, z_0, of f is an isolated singularity such that $f(z) = \frac{g(z)}{(z-z_0)^m}$, for some $g(z_0)\neq 0$ holomorphic at z_0.

$m=1 \implies$ a simple pole. $m=2 \implies$ double pole.

T: For an m^{th} order pole of f at c.

$$\operatorname*{Res}_{z=c}\{f(z)\} = \frac{1}{(m-1)!}\lim_{z\to c}\frac{d^{m-1}}{dz^{m-1}}\left[(z-c)^m f(z)\right]$$

T: Casorati–Weierstrass: In every neighbourhood of an isolated essential singularity of f, f takes values arbitrarily close to any given value infinitely often.

T: Picard's: In every neighbourhood of an isolated essential singularity f attains every given value with at most one exception, infinitely often.

T: f holomorphic in Ω a domain then

f has isolated zero order $m\in\mathbb{N}$ at $z_0\in\Omega$ $\iff$ $\frac{1}{f}$ has a pole of order m at $z\in\Omega$

⑩ Asymptotic Behaviour:

A neighbourhood of ∞ is some set $\{z\in\mathbb{C} \mid |z|>R\}$ some $R>0$.

$f \sim g$ as $z\to c \iff \lim_{z\to c}\frac{f(z)}{g(z)} = 1$

$f(z)=O(g(z))$ as $z\to c \iff (\exists\varepsilon>0)(\exists K>0)(\forall z\in D(c,\varepsilon))(|f(z)|<K|g(z)|)$

The function is bounded by the other in a neighbourhood of c.

$f(z)=o(g(z))$ as $z\to c \iff \lim_{z\to c}\frac{f(z)}{g(z)} = 0$.

Landau Rules.

$m,n\in\mathbb{Z}$, $z\to\infty$: $(1+O(z^m))(1+O(z^n)) = \begin{cases} 1+O(z^{\max\{m,n\}}), & m\leq 0 \text{ OR } n\leq 0 \\ 1+O(z^{m+n}), & m,n>0 \end{cases}$

$[1+O(z^m)]^{-1} = 1+O(z^m)$, $m\leq 0$

$m,n\in\mathbb{Z}$, $z\to 0$: $(1+O(z^m))(1+O(z^n)) = \begin{cases} 1+O(z^{\min(m,n)}), & m\geq 0 \text{ OR } n\geq 0 \\ 1+O(z^{m+n}), & m,n<0. \end{cases}$

$[1+O(z^m)]^{-1} = 1+O(z^m)$, $m\geq 0$.

(11) More General Contours:

T: Cauchy's Theorem: $C = \partial\Omega$ a simple closed contour with interior domain Ω and f holomorphic in Ω, f continuous on $\Omega \cup C$

$$\Rightarrow \oint_C f(z)\,dz = 0.$$

Integral Theorem $\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(t)\,dt}{(t-z)^{n+1}}$, where $f^{(0)} = f$, $f^{(1)} = f'$ etc.

T: Deformation: C_1 & C_2 contours in domain Ω, where f is holomorphic

- C_1 & C_2 start & end at the same point
- C_1 can be continuously deformed to C_2 without crossing non-holomorphic points of f

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

A bounded domain Ω is simply connected if $\text{comp}(\Omega)$ is connected.

(0) **T:** Laurent's in an Annulus: f holomorphic in $0 \le R_1 < |z - z_0| < R_2$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{2\pi i}\oint_{|t-z_0|=\rho} \frac{f(t)\,dt}{(t-z_0)^{n+1}}, \text{ some } R_1 < \rho < R_2.$$

T: $R_1 = \lim_{j\to\infty} \sup_{\ell \ge j} (a_{-\ell})^{\frac{1}{\ell}}$ **T:** $\frac{1}{R_2} = \lim_{j\to\infty} \sup_{\ell\ge j} (a_\ell)^{\frac{1}{\ell}}$

T: C a simple closed contour, interior domain Ω, f holomorphic in Ω except at finitely many isolated singularities $\{z_k\}_{k\in[n]}$, f is continuous on $\Omega \cup C$ except at $\{z_k\}$.

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k}\{f(z)\}$$

(12) Meromorphic Functions:

A function is meromorphic in a domain Ω if its only singularities in Ω are poles. If f & g are meromorphic in Ω then $\frac{1}{f}$, fg, $f+g$, $f\circ g$ are. (f meromorphic on range of g)

T: f meromorphic on domain Ω $\Rightarrow \frac{f'(z)}{f(z)}$ is meromorphic on Ω with simple poles at zeroes & poles of f.

$\frac{f'}{f}$ is the logarithmic derivative of f.

For f holomorphic & nonzero on contour C (simple closed) and meromorphic on the interior of C then we define $\mathcal{Z}(f;C) = \sum_{\text{zeroes of } f \text{ inside } C} [\text{order of zero}]$

$P(f;C) = \sum_{\text{poles of } f \text{ inside } C} [\text{order of pole}]$

T: f holomorphic & nonzero on simple closed contour C & meromorphic in its interior domain $\Rightarrow \frac{1}{2\pi i}\oint_C \frac{f'(z)\,dz}{f(z)} = \mathcal{Z}(f;C) - P(f;C)$

ROCHE'S THEOREM

T: f & g holomorphic on simple closed contour C, meromorphic on interior domain. $0 \le |g(z)| < |f(z)|$ on C.

$$\Rightarrow \mathcal{Z}(f+g;C) - P(f+g;C) = \mathcal{Z}(f;C) - P(f;C)$$

(13) Sequences of Functions:

Pointwise limit: $F(z) = \lim_{n\to\infty} F_n(z) \iff (\forall z)(\lim_{n\to\infty} F_n(z) = F(z))$.

For $\{F_n\}_{n\in\mathbb{N}}$ defined on S, $F_n \to F$ uniformly on S $\iff (\forall \varepsilon > 0)(\exists N > 0$ independent of $z)$ $(\forall n > N)(\forall z\in S)(|F_n(z) - F(z)| < \varepsilon)$

$\{F_n\}$ is uniformly Cauchy $\iff (\forall\varepsilon>0)(\exists N>0, z$ independent$)(\forall m,n > N)(\forall z\in S)(|F_m(z) - F_n(z)| < \varepsilon)$

T: $\{F_n(z)\}_{n\in\mathbb{N}}$ converge uniformly on $S \iff \{F_n(z)\}_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.

T: Weierstrass Safety Net: $\forall n\in\mathbb{N}$ F_n continuous on $S \subset \mathbb{C}$ and $F_n(z)$ converges uniformly on S

$\Rightarrow$
- $F(z) = \lim_{n\to\infty} F_n(z)$ exists.
- Is continuous on S.
- $\lim_{n\to\infty}\int_C F_n(z)\,dz = \int_C F(z)\,dz$ for C a contour of finite length
- [F_n converge uniformly on all compact subsets of domain Ω & $\forall n\in\mathbb{N}$ F_n is holomorphic $\Rightarrow$
  - F is holomorphic on Ω
  - F_n' converges uniformly to F' on all compact subsets of Ω.]

Series. $\sum_{n=1}^{\infty} F_n(z)$ converges uniformly on S if $\{\sum_{n=1}^{m} F_n(z)\}_{m\in\mathbb{N}}$ converges uniformly on S.

T: F_n continuous on S $\forall n\in\mathbb{N}$ and $F(z) = \sum_{n\ge1} F_n(z)$ converges uniformly on S. $\Rightarrow$ $F(z)$ continuous on S. $\int_C F(z)\,dz = \sum_{n\ge1}\int_C F_n(z)\,dz$ for C a contour of finite length

T: $\sum_{n\ge1} F_n$ converges uniformly on S $\iff (\forall\varepsilon>0)(\exists N>0$, independent of $z)(\forall m\ge k > N)$ $\left(\left|\sum_{n=k}^{m} F_n(z)\right| < \varepsilon\right)$ Cauchy Criterion.

T: $F(z) = \sum_{n\ge1} F_n(z)$ converges uniformly on all compact subsets of domain Ω & F_n holomorphic $\forall n\in\mathbb{N}$ $\Rightarrow$
- $F(z)$ holomorphic in Ω
- $F'(z) = \sum_{n\ge1} F_n'(z)$
- $F'(z)$ converges uniformly on all compact subsets of Ω.

T: Weierstrass M-Test: $(\forall z\in S\subset\mathbb{C})(|F_n(z)| \le M_n$, M_n independent of z and $\sum_{n\ge1} M_n$ converges$)$ $\Rightarrow \sum_{n\ge1} F_n(z)$ converges uniformly on S.

Applied to Taylor Series.

T: $\sum_{n\ge0} c_n (z-z_0)^n$ converges at $z=z_1$ $\Rightarrow$ converges absolutely in $D(z_0, |z_0 - z_1|)$

T: $\sum_{n\ge0} c_n(z-z_0)^n$ diverges at $z=z_1$ $\Rightarrow$ diverges $\forall z\in\text{comp}(D(z_0, |z_0 - z_1|))$.

Note that for both we don't know what happens on the circle, & out / inside respectively.

(14) Inverses on $\mathbb{C}$:

T: F holomorphic at z_0, $F'(z_0) \neq 0$ $\Rightarrow$
- $\exists M$ such that $\forall z\in D(F(z_0), M)$ $F(z) = x$ has exactly one solution, i.e. The inverse function $F^{-1}(x) = z$ exists.
- Further F^{-1} is holomorphic, $\frac{d}{dx}F^{-1}(x) = \frac{1}{F'(z)}$

Logarithm.

For $z\in\mathbb{C}\setminus\{0\}$ the principal value of complex $\log$ is

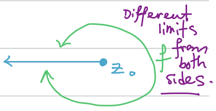
$$\log(z) = \log|z| + i\arg(z), \quad \arg(z)\in(-\pi, \pi].$$

(15) Analytic Continuation II:

T: $L(z)$ holomorphic in domain Ω, not containing 0.
Further $(\forall z\in\Omega)(L'(z)=z^{-1})$ AND $(\exists z_0\in\Omega)(e^{L(z_0)}=z_0)$
$\Rightarrow (\forall z\in\Omega)(\exp[L(z)]=z)$
A function satisfying the conditions above is a valid logarithm in the neighbourhood of z_0.

T: Ω_1 & Ω_2 domains, $z_0\in\Omega_1\cap\Omega_2\neq\emptyset$ (choose R st. this is the case)
L_1 & L_2 valid logarithms in $D(z_0,R)\subset\Omega_1\cap\Omega_2$
$\Rightarrow$ For every connected open subset of $\Omega_1\cap\Omega_2$
$\exists m\in\mathbb{Z}$ such that $L_1(z)=L_2(z)+2\pi i m$

A singularity z_0 of f such that f is discontinuous as you traverse the circle around it is a **branch point**. A cut in the plane drawn to avoid the point is a **branch cut**.



Powers.

For $z,c\in\mathbb{C}$ define $z^c=e^{c\log(z)}$, for any valid log.
The **principal value of z^c** is given by using the principle log.
$z^c=|z|^c e^{ic\arg(z)}$, $\arg(z)\in(-\pi,\pi]$.

T: For principle value of z^c
$|z|^{Re(c)}e^{-\pi|Im(c)|}\le|z^c|\le|z|^{Re(c)}e^{\pi|Im(c)|}$

T: $\frac{d}{dz}z^c=cz^{c-1}$

T: z^c has a branch cut on $\mathbb{R}_{\le 0}$

Branch Cuts.

The **Mellin transform** of Riemann integrable $f:(0,\infty)\to\mathbb{C}$ is
$\tilde f(s)=\int_0^\infty f(x)x^{s-1}dx$, $s\in\mathbb{C}$ is the frequency.

T: $Q(z)\overset{z\to\infty}{=}O(z^{-1})$ rational, poles at $\{z_k\}_{k\in[L]}\subset\mathbb{C}\setminus\{0\}$
$\Rightarrow$ [$0<Re(s)<1$ we have $\tilde Q(s)=-\frac{\pi}{\sin(\pi s)}\sum_{k=1}^{L}\underset{z=z_k}{Res}\{Q(-z)z^{s-1}\}$

T: $Q(z)\overset{z\to\infty}{=}O(z^{-2})$ rational with poles $\{z_k\}_{k\in[L]}\subset\mathbb{C}\setminus\{0\}$
$\Rightarrow \int_0^\infty Q(x)dx=\sum_{k=1}^{L}\underset{z=z_k}{Res}\{Q(-z)\log(z)\}$
$\Rightarrow \int_0^\infty Q(x)\log(x)dx=\frac{1}{2}\sum_{k=1}^{L}\underset{z=z_k}{Res}\{Q(-z)\log^2(z)\}$

T: $Q(z)\overset{z\to\infty}{=}O(z^{-2})$ rational $\{z_k\}_{k\in[L]}\subset\mathbb{C}\setminus[a,b]$
$\Rightarrow \int_a^b Q(x)dx=\sum_{k=1}^{L}\underset{z=z_k}{Res}\{Q(z)\log\left(\frac{z-b}{z-a}\right)\}$

T: $Q(z)\overset{z\to\infty}{=}O(z^{-3})$ rational $\{z_k\}_{k\in[L]}\subset\mathbb{C}\setminus[a,b]$
$\Rightarrow \int_a^b Q(x)\sqrt{(b-x)(x-a)}\,dx=-\pi\sum_{k=1}^{L}\underset{z=z_k}{Res}\{Q(z)\sqrt{(z-b)(z-a)}\}$

T: Fabry's: $\{\lambda_m\}$ strictly increasing sequence, $\lambda_m\in\mathbb{N}_0$, $\frac{\lambda_m}{m}\xrightarrow{m\to\infty}\infty$
and $F(z)=\sum_{m\ge0}a_m z^{\lambda_m}$ has radius of convergence 1.
$\Rightarrow F(z)$ cannot be analytically continued beyond $|z|=1$.

Riemann Zeta.

Any series of form $\sum_{n\ge1}\frac{a_n}{n^s}$, $s\in\mathbb{C}$, a_n s independent is a **Dirichlet series**.

T: $A(s)=\sum_{n\ge1}\frac{a_n}{n^s}$ converges for $s=s_0$ $\Rightarrow$
- Converges uniformly $\forall\delta\in(0,\frac{\pi}{2})$ $|\arg(s-s_0)|\le\frac{\pi}{2}-\delta$
- $A(s)$ is well defined in $Re(s)>Re(s_0)$ & is holomorphic there. $A'(s)=\sum_{n\ge1}\frac{a_n\log(n)}{n^s}$

The **Riemann Zeta Function** is defined as
$\zeta(s)=\sum_{n\ge1}\frac{1}{n^s}=\sum_{n\ge1}\exp[-s\log(n)]$
for $s\in\mathbb{C}$ where this converges. We need another definition for the continuation.
Denote **$\mathbb{P}\subset\mathbb{N}$ the set of all primes**, $\zeta(s)=\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}$

T: ζ is holomorphic everywhere except $s=1$, the residue at $s=1$ is 1.

Gamma Function.

T: $(n\in\mathbb{N})(a\in(0,n))(G_n(z,a)=\int_a^n e^{-t}t^{z-1}dt$ is entire$)$
The **incomplete Gamma** is $\Gamma(z,a)=\int_a^\infty e^{-t}t^{z-1}dt$, $a>0$

T: $\Gamma(z,a)$ is entire for $a>0$.

The **gamma function is** defined as
$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt=\int_0^1 e^{-t}t^{z-1}dt+\Gamma(z,1)$
For $z\in\mathbb{C}$ where the integral exists. By analytic continuation else where.

T: $\Gamma(z)$ is meromorphic in $\mathbb{C}$. Γ has only simple poles at $z=0,-1,-2,\dots$ with $\underset{z=-n}{Res}\{\Gamma(z)\}=\frac{(-1)^n}{n!}$

T: $|\Gamma(z)|\le\Gamma(Re\{z\})$, $Re(z)>0$

T: $\Gamma(z+1)=z\Gamma(z)$, $z\in\mathbb{C}$

T: $n\in\mathbb{N}_0\Rightarrow\Gamma(n+1)=n!$ T: $\Gamma(\frac{1}{2})=\sqrt{\pi}$

T: $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$, $z\in\mathbb{C}\setminus\mathbb{Z}$

T: $\Gamma(z)\neq0$ $\forall z\in\mathbb{C}\setminus(-\mathbb{N}_0)$ T: $\frac{1}{\Gamma(z)}$ is entire

T: $\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2})$

The **Beta function** is $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt$, $Re(u),Re(v)>0$.

T: $B(u,v)=B(v,u)$ T: $B(u,v)=\int_0^\infty\frac{x^{u-1}dx}{(1+x)^{u+v}}$, $Re(u),R(v)>0$

T: $B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ Most general β definition too.

Relating ζ & Γ

$R>0$ we define the loop contour integral as $\int_{-R}^{(0+)} f(z)\,dz$ as a contour integral over a simply closed contour encircling the origin & cutting the negative real axis at $-R$ only.

The Hankel loop contour integral is $\int_{-\infty}^{(0+)} f(z)\,dz$

T: For Q holomorphic on $\mathbb{R}^+$, if Q is singular at 0 it is a pole, $\exists \rho$ such that there are no other singularities of Q within ρ of the positive real axis

$$\Rightarrow \frac{1}{2\pi i}\int_{-\infty}^{(0+)} z^{s-1}Q(-z)\,dz = \frac{\sin(\pi s)}{\pi}\int_0^\infty t^{s-1}Q(t)\,dt.$$

T: $\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt$, $Re(s)>1$.

T: $\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{(0+)} \frac{z^{s-1}}{e^{-z}-1}\,dz$, $s\notin\mathbb{N}$

T: Reimann Relation: $s\neq 1$, $\zeta(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1-s)\zeta(1-s)$

(16) Harmonic Analysis:

Fourier.

The n^{th} fourier coefficient for f absolutely integrable & cnts, $L>0$ is $\tilde{f}_n = \frac{1}{L}\int_0^L f(t)e^{2\pi i\frac{nt}{L}}\,dt$

T: $L>0$, f continuous & absolutely integrable on $(0,L)$ with $\sum_{n=-\infty}^{\infty}|\tilde{f}_n|<\infty \Rightarrow (\forall t\in(0,L))(f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\tilde{f}_n e^{2\pi i\frac{nt}{L}})$.

T: $\tilde{f}_n + \tilde{f}_{-n} = \frac{2}{L}\int_0^L f(t)\cos[2\pi\frac{nt}{L}]\,dt$ and

$-i(\tilde{f}_n - \tilde{f}_{-n}) = \frac{2}{L}\int_0^L f(t)\sin[2\pi\frac{nt}{L}]\,dt$.

T: f rational with poles $\{z_k\}_{k\in[L]}\notin\mathbb{Z}$ & $f(z)\overset{n\to\infty}{=}O(z^{-2})$

$$\Rightarrow \sum_{l=-\infty}^{\infty} f(l) = -\pi\sum_{j=1}^{L}\text{Res}_{z=z_j}\{f(z)\frac{1}{\tan(\pi z)}\}$$

T: f rational poles at $\{z_k\}_{k\in[L]}\notin\mathbb{Z}$, $f(z)\overset{n\to\infty}{=}O(z^{-1})$

$$\Rightarrow \sum_{l=-\infty}^{\infty}(-1)^l f(l) = -\pi\sum_{j=1}^{L}\text{Res}_{z=z_j}\{\frac{f(z)}{\sin(\pi z)}\}$$

For a function f Riemann integrable on every finite $[a,b]\subset(A,B)\subset\mathbb{R}$, $A,B\in\mathbb{R}\cup\{\pm\infty\}$, $c\in(A,B)$

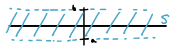
$$\Rightarrow \int_A^B f(t)\,dt = \lim_{a\to A}\int_a^c f(t)\,dt + \lim_{b\to B}\int_c^b f(t)\,dt$$

is the improper Riemann integral.

The Fourier transform of $f:\mathbb{R}\to\mathbb{C}$ is $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$, $k\in\mathbb{C}$ complex frequency

The inverse Fourier Transform is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)e^{-ikx}\,dk$.

f must be continuous & Riemann integrable obviously.



T: $a<b\in\mathbb{R}$, f holomorphic inside $S=\{z\mid Im(z)\in(a,b)\}$, continuous on $\bar{S}=S\cup\partial S$, $\lim_{|x|\to\infty}\max_{y\in[a,b]}|f(x+iy)|=0$

$$\Rightarrow (\forall z_1,z_2\in\bar{S})\left(\int_{-\infty}^{\infty}f(x+z_1)\,dx = \int_{-\infty}^{\infty}f(x+z_2)\,dx\right)$$

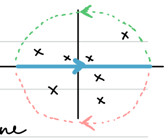
Translation

T: $\varphi<\vartheta\in(-\pi,\pi]$, f holomorphic in $S=\{re^{ix}\in\mathbb{C}\mid r>0, x\in(\varphi,\vartheta)\}$, cnt's on $\bar{S}=S\cup\partial S$, & $\lim_{r\to\infty}\max_{x\in[\varphi,\vartheta]}|rf(re^{ix})|=0$

$$\Rightarrow (\forall x_1,x_2\in[\varphi,\vartheta])\left(\int_0^\infty f(re^{ix_1})\,dr = \int_0^\infty f(re^{ix_2})\,dr\right)$$

Rotation

T: $Q(x) = \frac{\text{polynomial degree } m}{\text{polynomial degree } n}$, $m\le n-1$



$\{z_k^{(\pm)}\}_{k=1,\dots,L\pm}$ poles on the upper / lower half plane respectively

$$\Rightarrow \int_{-\infty}^{\infty}e^{itx}Q(x)\,dx = \begin{cases} 2\pi i\sum_{k=1}^{L+}\text{Res}_{z=z_k^{(+)}}\{e^{itz}Q(z)\}, & t\ge 0\\ -2\pi i\sum_{k=1}^{L-}\text{Res}_{z=z_k^{(-)}}\{e^{itz}Q(z)\}, & t\le 0\end{cases}$$

The Cauchy principal value integral of f, for $a,b,c\in\mathbb{R}$.

- $\fint_{-\infty}^{\infty}f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx$
- $\fint_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon}f(x)\,dx + \int_{c+\varepsilon}^{b}f(x)\,dx\right)$

Importantly the limits are taken simultaneously.

T: Jordans Lemma: $S=\{Re^{it}\mid t\in[0,\pi]\}$, $R>0$, $k>0$, f continuous on S $\Rightarrow \left|\int_S e^{ikz}f(z)\,dz\right| \le \frac{\pi}{k}(1-e^{-kR})\max_{z\in S}|f(z)|$.

T: Kronig-Kramer: f holomorphic in closed upper half plane $\lim_{\substack{z\to\infty\\ Im(z)\ge 0}} f(z)=0$, $f=u+iv$, u & v real functions

$$\Rightarrow (\forall y\in\mathbb{R})\left(u(y) = \frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{v(x)\,dx}{x-y} \;\&\; v(y) = -\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{u(x)\,dx}{x-y}\right)$$

A relation similar to the Cauchy-Riemann relations.

T: Sokhotski-Plemelj: f continuous absolutely integrable on $(-\infty,\infty)$

$$\Rightarrow (\forall y\in\mathbb{R})\left(\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-y\pm i\varepsilon} = \fint_{-\infty}^{\infty}\frac{f(x)\,dx}{x-y}\mp\pi i f(y)\right)$$

Harmonic Functions.

T: For a holomorphic $f(z)=\phi(x,y)+i\psi(x,y)$, ϕ & ψ real valued $\Rightarrow$ ϕ & ψ have pure & mixed partial derivatives with respect to x & y of all orders. Moreover the order of derivatives can be interchanged.

A real valued solution $g(x,y)$ to the laplace equations $\Delta g(x,y) = \frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 0$ is a harmonic func.

T: The real & imaginary parts ϕ,ψ of a holomorphic $f=\phi+i\psi$ both satisfy the Laplace equations.

ie. $\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$ & $\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi = 0$.

Solutions of the Laplace equations that are connected by the Cauchy-Riemann relations are called conjugate harmonic functions.

T: ϕ a harmonic function on a simply connected domain Ω $\Longrightarrow$ $\exists f$ holomorphic on Ω, $\phi(x,y) = Re[f(x,y)]$.

Note that f will not be unique.

Any holomorphic function consists of a set of conjugate harmonic functions, this tells us that on a simply connected domain this relation is invertable.

Denote $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \Delta\phi$.

T: $(\forall \mu,\nu\in\mathbb{C})(\Delta(\mu\phi+\nu\psi)=\mu\Delta\phi+\nu\Delta\psi)$

Laplace equations are linear.

So linear combinations of solutions are solutions.

We usually solve laplace equations on a domain with some boundary conditions. C a differentiable contour & $g_0: C\to\mathbb{R}$ a differentiable function along C. Then a Dirichlet boundary condition for a DE in ϕ is
$\phi(x,y)=g_0(x,y)\ \forall(x,y)\in C$.

A Neumann boundary condition for a DE in ϕ is $\hat n(x,y)\cdot\nabla\phi(x,y) = g_0(x,y)\ \forall (x,y)\in C$ where $\hat n$ is the unit normal vector of C at (x,y), $\cdot$ is the inner product & $\nabla\phi = \frac{\partial\phi}{\partial x}\hat i + \frac{\partial \phi}{\partial y}\hat j$. $\hat i=(1,0)$, $\hat j=(0,1)$

T: Ω a domain, $C=\partial\Omega$ differentiable contour, $g_0: C\to\mathbb R$ differentiable function on C. $f=\phi+i\psi$ with both ϕ & ψ holomorphic on Ω. $\hat n$ unit normal, $\hat t$ unit tangent of C at (x,y); with $\det(\hat n,\hat t)=1$

THEN $\left[\begin{array}{l}\forall(x,y)\in C \\ \phi(x,y)=g_0(x,y)\end{array}\right] \iff \left[\begin{array}{l}(\forall x,y\in C)(-\hat n\cdot\nabla\psi(x,y) = \hat t\cdot\nabla g_0(x,y)) \\ \& \ (\exists(x_0,y_0)\in C)(\phi(x_0,y_0)=g_0(x_0,y_0))\end{array}\right]$.

For a holomorphic $f(z)$, $z=x+iy$, $f=\phi+i\psi$ we call ϕ the potential, lines of constant ϕ are equipotentials. ψ is the stream function, lines of constant ψ are called streamlines. & f is called the complex potential.

T: Equipotentials & stream lines always intersect at $\perp$.

Conformal Maps.

A holomorphic function f on domain Ω, such that $(\forall z\in\Omega)(f'(z)\neq 0)$ is a conformal map.

These are useful maps because under a conformal map the straight line joining z to $z+\delta z$ is

- translated by $f(z)-z$ — dilated by $|f'(z)|$
- rotated by $\arg(f'(z))$ & angles between tangents to curves are preserved. Also gives local invertibility of f.

T: Consider a bijective conformal map p from domain Ω to w-plane. ie. $w=p(z) \iff z=p^{-1}(w)$.

Also consider simply connected domain Z in complex z plane.

$\Longrightarrow \left[\begin{array}{l}\phi:\Omega\to\mathbb{C} \text{ is a solution to} \\ \Delta\phi=0 \text{ in } \Omega\end{array} \iff \begin{array}{l}\phi\circ p \text{ is a solution} \\ \text{to } \Delta(\phi\circ p) \text{ in } Z\end{array}\right]$

Laplace equation.

T: f entire $\Rightarrow$ $f(\mathbb C)$ is dense in $\mathbb C$.

T: No conformal maps exist from $\mathbb C$ to

- bounded domain
- Exterior of bounded domain
- half plane.

Two sets, Ω a domain, Φ a set such that there exists a bijective conformal map $f:\Omega\to\Phi$ are called conformally equivalent.

A bijective conformal map $f:\Omega\to\Omega$ is a conformal automorphism.

T: Any two simply connected domains, such that neither are the entire $\mathbb C$, are conformally equivalent.

Conformal Automorphisms of $\overline{\mathbb C}$.

T: The only conformal automorphisms of $\mathbb C$ are linear maps: $f(z)=az+b$, $a,b\in\mathbb C$, $a\neq 0$.

$\overline{\mathbb C} = \mathbb C\cup\{\infty\}$. There are arithmetic rules...

$\forall z\in\mathbb C$ $z+\infty = \infty+z = \infty$, $\frac{z}{\infty}=0$

$\forall z\in\overline{\mathbb C}\setminus\{0\}$ $z\infty = \infty z = \infty$

$0/0$, ∞/∞, 0∞, $\infty\pm\infty$ are NOT well defined.

T: Linear maps are conformal automorphisms of $\overline{\mathbb C}$.

T: $z\longmapsto\frac{1}{z}$ is a conformal automorphism of $\overline{\mathbb C}$

T: Mobius transforms are the only conformal maps of $\overline{\mathbb C}$

A mobius transform is a meromorphic function
$T\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}, z\right) = \frac{az+b}{cz+d}$, $ad-bc\neq 0$.

T: $f(z)=z^{-1}$ bijectively maps

- $\partial D(z_0,r) \longleftrightarrow \partial D\left(\frac{z_0^*}{|z_0|^2-r^2}, \frac{r}{||z_0|^2-r^2|}\right)$, $|z_0|\neq r>0$
- $\partial D(z_0,|z_0|) \longleftrightarrow \{z\in\mathbb C \mid Re(z_0 z)=\frac{1}{2}\}$, $z_0\in\mathbb C\setminus\{0\}$
- $\{z\in\mathbb C \mid Re(e^{i\theta_0}z)=0\} \longleftrightarrow \{z\in\mathbb C \mid Re(e^{-i\theta_0}z)=0\}$, $\theta_0\in\mathbb R$.

T: A mobius transformation is a composition of rotations, dilations, translations & inversions.

T: $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{C}$, $ad - bc \neq 0$

$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, z\right)\right) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, z\right)$$

T: Any mobius transformation maps circles & straight lines onto circles & straight lines.

T: Set of all mobius transformations is a group with composition as the action.

$SL(2,\mathbb{C})$ is the group of complex 2×2 maticies with determinant 1.

$H_+ = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

T: Real parameter mobius transforms with $ad - bc > 0$ are conformal automorphisms of of $H_+ \cup \{\infty\}$.

A mobius transform such that $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ is a modular transformation. The group of these transformations is called the modular group.

Fundamental set ?

T: f holomorphic on simple closed contour $C \subset \Omega$

Ω a domain

$\Longrightarrow \forall z_0 \in$ interior of $\hat{C} = \{ \frac{z_0^{-1}}{z} \mid z \in C \}$

$$\int_C f(z)\,dz \overset{z = \frac{1}{w - z_0}}{=} \oint_{\hat{C}} f\left(\frac{1}{w - z_0}\right) \frac{-dw}{(w - z_0)^2}$$

$$= \oint_{\hat{C}} f\left(\frac{1}{w - z_0}\right) \frac{dw}{(w - z_0)^2}$$

T: $Q(z)$ rational $\{z_k\}_{k \in [L]}$, $R > 0$ large enough so that all poles lie in open disc $D(0, R)$

$$\Rightarrow \sum_{k=1}^{L} \operatorname*{Res}_{z = z_k} \{ Q(z) \} = \operatorname*{Res}_{w=0} \left\{ \frac{1}{w^2} Q\left(\frac{1}{w}\right) \right\}$$

The residue at infinity is defined by

$$\operatorname*{Res}_{w = \infty} \{ Q(w) \} = -\operatorname*{Res}_{w=0} \left\{ \frac{1}{w^2} Q\left(\frac{1}{w}\right) \right\}$$

T:
$$\int_a^b Q(x)\,dx = \sum_{k=0}^{L} \operatorname*{Res}_{z = z_k} \left\{ Q(z) \log\left(\frac{z - b}{z - a}\right) \right\} - \operatorname*{Res}_{w=0} \left\{ \frac{Q(w^{-1})}{w^2} \log\left(\frac{1 - bw}{1 - aw}\right) \right\}$$

T:
$$\int_a^b Q(x) \sqrt{(b - x)(x - a)}\,dx = -\pi \sum_{k=1}^{L} \operatorname*{Res}_{z = z_k} \left\{ Q(z) \sqrt{(z - b)(z - a)} \right\} - \operatorname*{Res}_{w=0} \left\{ \frac{Q(w^{-1})}{w^3} \sqrt{(1 - bw)(1 - aw)} \right\}$$